ON A DYNAMICAL METHOD FOR INCREASING STABILITY OF A RAPIDLY ROTATING SYMMETRICAL GYROSCOPE

(OB ODNOM DINAMICHESKOM METODE POVYSHENIIA USTOICHIVOSTI BYSTROVRASHCHAIUSHCHEGOSIA SIMMETRICHNOGO GIROSKOPA)

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There are instances in mechanics when the stability of certain motions of a mechanical system can be significantly improved by application of external time-varying forces, while some normally unstable motions can even be completely stabilized. An example is a pendulum with an oscillating support. In this case the stability of the lower position of equilibrium is considerably increased with respect to external disturbances, while for certain conditions the upper position of the pendulum can become stable [1]. On the basis of this principle accelerators with rigid focusing are constructed which increase the stability of orbital motions of charged particles [2].

It will be shown below that the stability of a rapidly rotating gyroscope can be increased by an analogous method.

The rotation of a heavy symmetrical gyroscope with respect to its pole can be described by a Hamiltonian of the following form:

$$H_0 = \frac{p_\varphi^2}{2I_3} + \frac{1}{2I}\left[p_\theta^2 + \left(\frac{p_\psi - p_\varphi \cos\theta}{\sin\theta}\right)^2\right] + Mgl\cos\theta \qquad (1)$$

Here I_3, I are the principal moments of inertia of the gyroscope relative to the pole, l is the distance from the center of masses to the pole.

It is known that the natural oscillation of the gyroscope will be stable if the coefficient of gyroscopic stability

$$\sigma_0 = \sqrt{1 - \frac{4IMgl}{I_3^2\omega_3^2}} \qquad (2)$$

is bounded in the region $1 \geqslant \sigma_0 \geqslant 0$, and also that the higher σ_0 the more stable will be the natural oscillation with respect to external disturbances [3].

It can be seen from Formula (2) that for the highest stability of the gyroscope it is necessary to make the quantity $IMgl/I_3^2\omega_3^2$ as small as possible. Should this possibility be exhausted, further increase in stability can be obtained by vibration of the pole.

Thus, let the pole vertical vibration be prescribed by $z_0 = F(t)$, where we will consider the function $F(t)$ to be almost periodic, which can be expressed in the form of a finite sum of periodic functions with non-coincident periods:

$$z_0 = \sum_{\nu} \sum_{n \neq 0} a_{\nu n} \exp(i\omega_\nu nt)$$

Let us consider the motion of a gyroscope relative to the system of coordinates referred to its pole. However, because of vibration this system will not be inertial, and it is therefore necessary to add the potential of inertia forces to the Hamiltonian, which will then be of the form

$$H = H_0 - M \cos\theta \sum_{\nu, n \neq 0} \omega_\nu^2 a_{\nu n} n^2 \exp(i\omega_\nu nt) \tag{3}$$

Let ω be the highest of the frequencies ω_ν. Introducing nondimensional time $\tau = \omega t$, the Hamiltonian now becomes

$$H' = \varepsilon [H_0 - Ml\omega^2 \cos\theta \sum_{\nu} \sum_{n \neq 0} \varkappa_\nu^2 a_{\nu n} n^2 \exp(i\varkappa_\nu n\tau) \qquad \left(\varepsilon = \frac{1}{\omega}, \quad \varkappa_\nu = \frac{\omega_\nu}{\omega} \leqslant 1\right) \tag{4}$$

The exact solution of Hamilton's equations stemming from (4) is impossible. It is therefore necessary to use an approximate method which consists of the following [4, 5].

With the aid of function $S(q, P; \tau)$ we change the variables p, q to the variables P, Q, assuming

$$p = \partial S / \partial q, \qquad Q = \partial S / \partial P$$

The Hamiltonian H^* in the new variables will be related to the Hamiltonian H in the original variables by

$$\varepsilon H^* = \varepsilon H + \partial S / \partial \tau \tag{5}$$

Seeking S such that the Hamiltonian in the new variables be explicitly independent of time, the equality (5) then transforms into an equation for determination of the function S. We have

$$\frac{\partial S}{\partial \tau} + \varepsilon H\left(\frac{\partial S}{\partial q}, q; \tau\right) = \varepsilon H^*\left(P, \frac{\partial S}{\partial P}; \varepsilon\right) = \varepsilon H_1^*\left(P, \frac{\partial S}{\partial P}\right) + \varepsilon^2 H_2^*\left(P, \frac{\partial S}{\partial P}\right) + \cdots \quad (6)$$

We seek the solution of Equation (6) in the form of the series

$$S = S_0 + \varepsilon S_1 + \varepsilon^2 S_2 \quad (7)$$

Here $S_0 = Pq$ is the identity transformation. Substituting (7) into (6), expanding in powers of ϵ and equating the coefficients of equal powers of ϵ, we obtain the following chain of equations for successive approximations:

$$\frac{\partial S_1}{\partial \tau} + H(P, q; \tau) = H_1^*(P, q), \qquad \frac{\partial S_2}{\partial \tau} + \frac{\partial H}{\partial P}\frac{\partial S_1}{\partial q} = \frac{\partial H_1^*}{\partial q}\frac{\partial S_1}{\partial P} + H_2^*(P, q) \quad (8)$$

Since H is an almost periodic function of τ, i.e.

$$H(P, q; \tau) = \sum_{n, \nu}{}' H_{n\nu} \exp(in\varkappa_\nu \tau) \quad (9)$$

the solution of (8) will be

$$H_1^* = \sum_{\nu}{}' H_{\nu_0}, \qquad S_1 = i \sum_{\nu, n \neq 0}{}' \frac{H_{n\nu}}{n\varkappa_\nu} \exp in\varkappa_\nu \tau$$

Substituting the result of the first approximation into the equation of second approximation (9), one can analogously find S_2 and H_2* etc., as series in powers of ϵ.

Applying this method to the Hamiltonian (4), in the third approximation we will obtain

$$H = H_0 + G\sin^2 Q \qquad \left(G = \frac{M^2 l^2}{2I}\sum_{\nu, n} a_{n\nu}\, a_{n\nu}\, \omega_\nu{}^2\right) \quad (10)$$

$$S = P_\varphi \varphi + P_\psi \psi + P_\theta \theta - iMl\cos\theta \sum_{\nu, n\neq 0}{}' a_{\nu n}\omega_\nu n \exp^{in\omega_\nu t} +$$
$$+ \frac{Ml}{I} P_\theta \cos\theta \sum_{\nu, n \neq 0} a_{\nu n} \exp^{in\omega_\nu t} + \cdots \quad (11)$$

The original variables p_ψ, p_ϕ, p_θ; ψ, ϕ, θ are related to the new ones by the following formulas:

$$p_\psi = \frac{\partial S}{\partial \psi} = P_\psi, \qquad p_\varphi = \frac{\partial S}{\partial \varphi} = P_\varphi, \qquad p_\theta = \frac{\partial S}{\partial \theta} = P_\theta + \varepsilon \frac{\partial S_1}{\partial \theta} + \cdots$$
$$\Psi = \frac{\partial S}{\partial P_\psi} = \psi, \qquad \Phi = \frac{\partial S}{\partial P_\varphi} = \varphi, \qquad Q = \frac{\partial S}{\partial P_\theta} = \theta + \varepsilon\frac{\partial S_1}{\partial P_\theta} + \cdots$$

For further investigations it is convenient to change from the

Hemiltonian form of equations to the Lagrangean form, since the initial conditions are formulated for the coordinates and the velocities. To the Hamiltonian (10) there is a corresponding Lagrangean

$$L = \frac{1}{2} I_3 [\dot{\Psi} \cos Q + \dot{\Phi}]^2 + \frac{1}{2} I [\dot{\Psi}^2 \sin^2 Q + \dot{Q}^2] - Mgl \cos Q - G \sin^2 Q$$

Let us investigate the stability of the gyroscope motion with the initial conditions

$$\begin{array}{lll} \Phi_0 = \varphi_0 = 0, & \Psi_0 = \psi_0 = 0, & Q_0 = \theta_0 = 0 \\ \dot{\Phi}_0 = \dot{\varphi}_0 = \omega_3, & \dot{\Psi}_0 = \dot{\psi}_0 = 0, & \dot{Q}_0 = \dot{\theta}_0 = 0 \end{array} \qquad (12)$$

Here ω_3 is the natural frequency of rotation; also $Q_0 = \theta_0 = 0$, $\dot{Q}_0 = \dot{\theta}_0 = 0$ are satisfied in any case for this approximation.

The Lagrange equations for the three Euler coordinates are

$$\frac{\partial L}{\partial \dot{\Phi}} = \text{const}, \qquad \dot{\Psi} \cos Q + \dot{\Phi} = \omega_3 = \text{const} \qquad (13)$$

$$\frac{\partial L}{\partial \dot{\Phi}} = \text{const}, \qquad I_3\omega_3 \cos Q + I\dot{\Psi} \sin^2 Q = aI = \text{const} \qquad (14)$$

$$\frac{d}{dt}\frac{dL}{d\dot{Q}} - \frac{\partial L}{\partial \dot{Q}} = 0, \qquad \ddot{Q} + \frac{I_3\omega_3}{I} \dot{\Psi} \sin Q - \dot{\Psi}^2 \sin Q \cos Q - \frac{Mgl}{I} \sin Q + \frac{2G}{I} \sin Q \cos Q = 0 \qquad (15)$$

Further

$$a = \frac{I_3\omega_3}{I} \cos Q_0 + \dot{\Psi}_0^2 \sin^2 Q_0 = \frac{I_3\omega_3}{I}$$

Introducing the notation

$$\alpha_0 = \frac{1}{2} \frac{I_3\omega_3}{I}, \qquad \beta = \frac{Mgl}{I}$$

and eliminating $\dot{\psi}$ from Equation (15) we obtain

$$\dot{Q}^2 + 4\alpha_0^2 \frac{(1 - \cos Q)^2}{\sin^2 Q} + 2\beta \cos Q + \frac{2G}{I} \sin^2 Q = 2\beta \qquad (16)$$

Let us introduce $u = \cos Q$; then Equation (16) becomes

$$\dot{u}^2 = 2\beta (1 - u^2)(1 - u) - 4\alpha_0^2 (1 - u)^2 - \frac{2G}{I} (1 - u^2)^2 = P_4(u) \qquad (17)$$

As in the case of no vibration, the Lagrange equation (16) has the following solution:

$$\varphi = \omega_3 t, \qquad \Psi = 0, \qquad Q = 0 \qquad \text{or } u = 1$$

This solution corresponds to the natural rotation [spin] of the gyroscope without nutation and precession, i.e. with the retention of the given direction of rotation, this form of rotation being widely utilized in technology. This motion must be stable with respect to the external disturbances.

The solution $u = 1$, $u = 0$ will be stable if

$$\frac{\partial^2 P_4(u)}{\partial u^2} < 0 \qquad \text{for } u = 1 \tag{18}$$

The value of the second derivative may be taken as the measure of stability. Equation (18) gives

$$\sigma = \sqrt{1 - \frac{\beta}{\alpha_0^2 + 2G/I}} = \sqrt{1 - \frac{4MglI}{I_3^2\omega_3^2 + 8G}}$$

or, if the so-called coefficient of gyroscopic stability is introduced [3, p. 152], which is compared with that of the gyroscope without vibration [2], then it will be clear that $\sigma > \sigma_0$ for any condition, i.e. the vibrations of the stationary point of the gyroscope (pole) increase its resistance to external disturbances.

We state without proof that the smaller the amplitude of vibration and the higher its frequency the greater is effectiveness of this method for increasing stability.

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